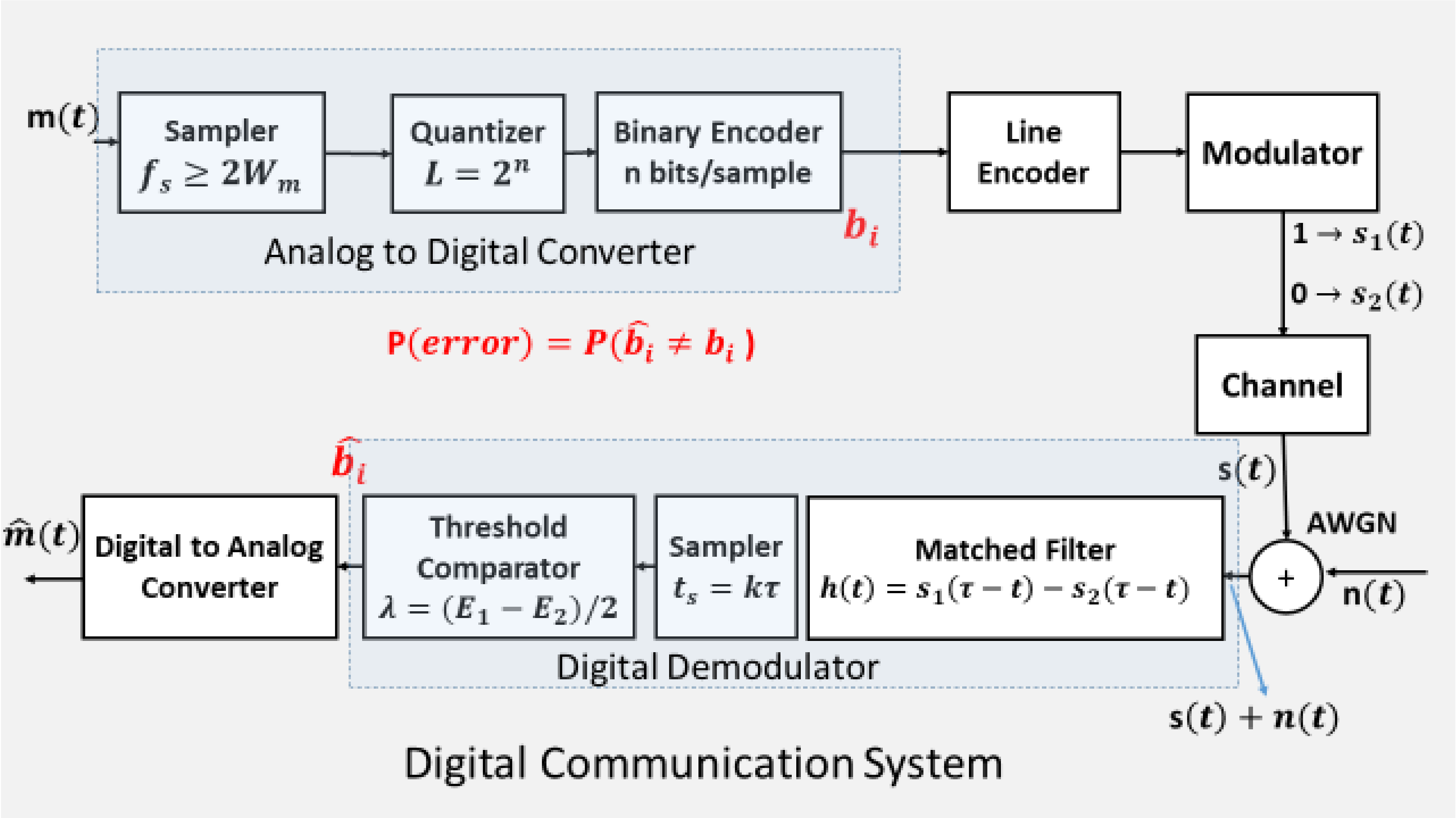
Matched Filter and Performance of the Optimum Receiver



Theorem on the Optimum Binary Receiver

Consider a binary communication system, corrupted by AWGN with power spectral density $N_0/2$, where the equally probable binary digits 1 and 0 are represented by the signals $s_1(t)$ and $s_2(t)$, respectively. The transmission time for each signal is τ sec. The optimum receiver elements, i.e., the elements that minimize the receiver probability of error are given by

Impulse response of the matched filter: $\boldsymbol{h(t) = s_1(\tau - t) - s_2(\tau - t)}$, $0 \leq t \leq \tau$

Optimum sampling time: $\boldsymbol{t_s = \tau}$

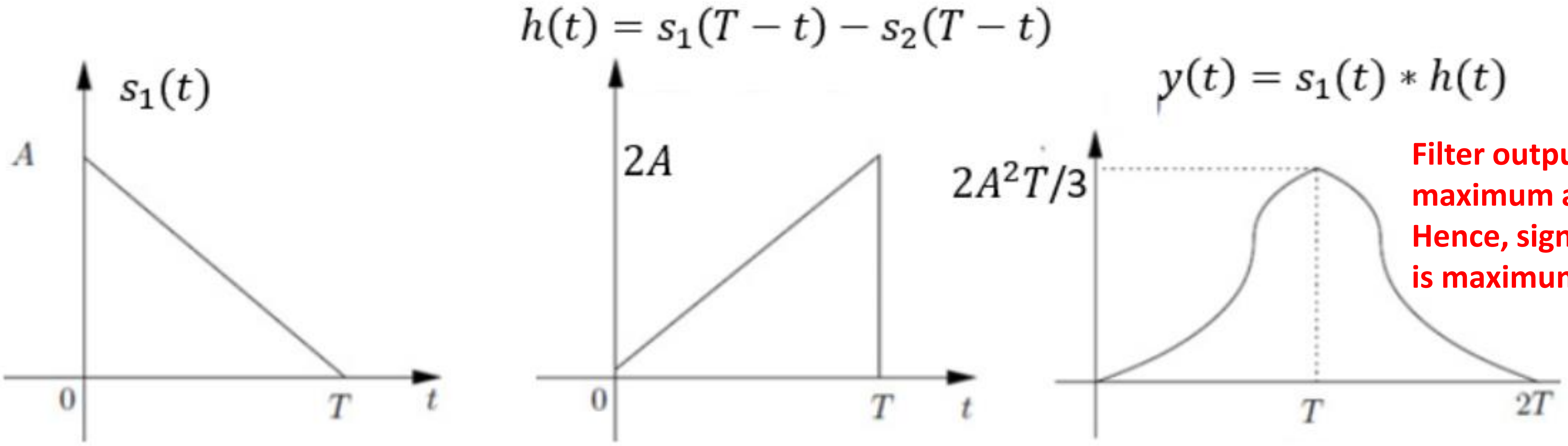
Optimum threshold of comparator: $\lambda^* = \frac{1}{2}(\boldsymbol{E_1 - E_2})$, $E_k = \int_0^{\tau} (s_k(t))^2 dt$, $k = 1,2$

When these elements are used, the system minimum probability of error is

$$\boldsymbol{P_b^* = Q\left(\sqrt{\frac{\int_0^{\tau} (s_1(t) - s_2(t))^2 dt}{2N_0}}\right)}$$

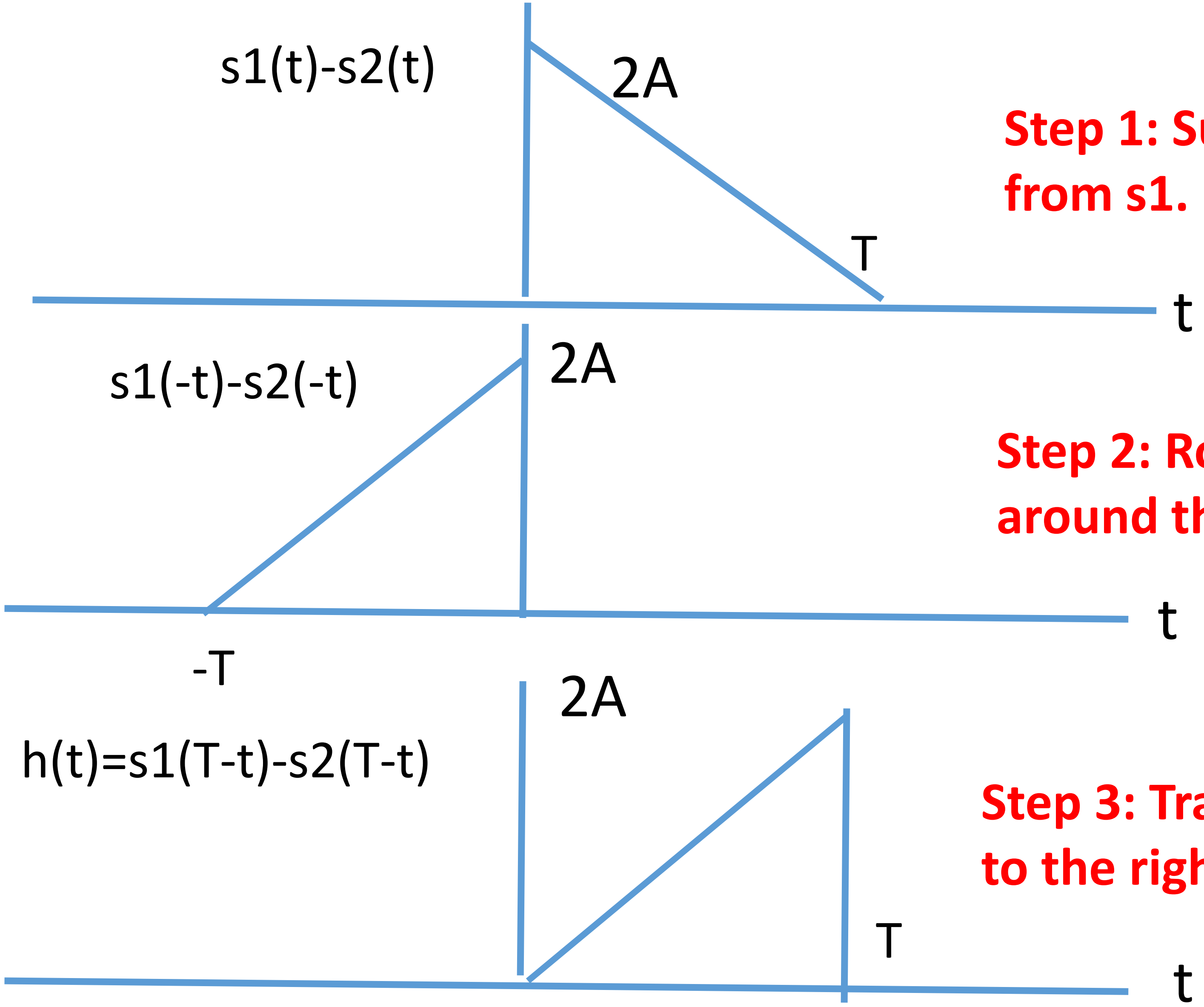
Output of a Matched Filter

Example 2: The next figure shows a signaling scheme where $s_2(t) = -s_1(t)$. The impulse response of the matched filter is $h(t) = s_1(T-t) - s_2(T-t)$. The figure shows the filter output when $s_1(t)$ is applied to the filter. Note that the output attains its maximum value at time t=T, which is the sampling time chosen to maximize the output SNR.



Filter output is maximum at t=T. Hence, signal power is maximum

Matched Filter Derived from Signals



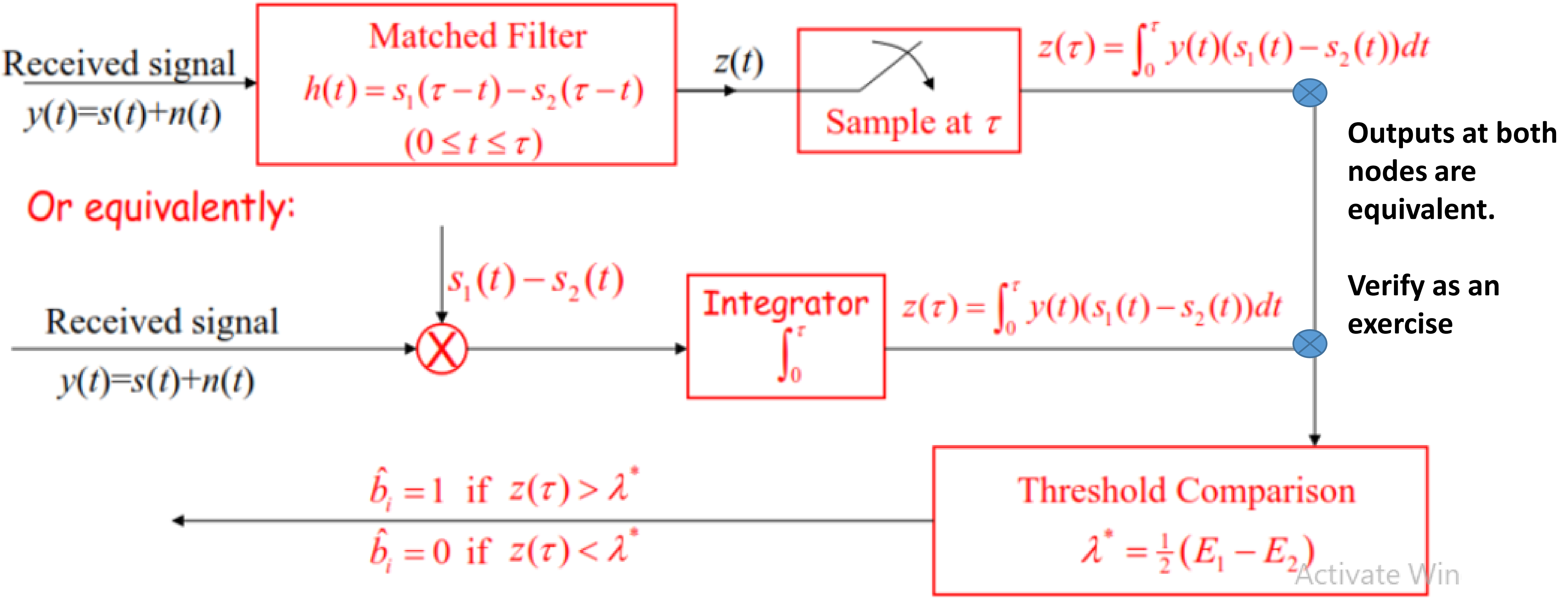
Step 1: Subtract s2 from s1.

Step 2: Rotate around the y-axis

Step 3: Translate to the right by T

Equivalent Implementations of the Optimum Receiver

The structure of the optimum receiver is depicted in the figure below. Note that the receiver can be implemented in terms of the matched filter and, equivalently, in terms of a correlator (a multiplier followed by an integrator).



Example: Antipodal Binary Transmission

Let us consider a digital binary communication system where bits 1 and 0 are represented by the signals $s_1(t)$ and $-s_1(t)$, respectively.

For this case, $E_1 = E_2 = E = \int_0^\tau (s_1(t))^2 dt$

Therefore, the threshold is $\boldsymbol{\lambda^* = (E_1 - E_2) = 0}$

The probability of error is:

$$P_b^* = Q\left(\sqrt{\frac{\int_0^\tau (s_1(t) - s_2(t))^2 dt}{2N_0}}\right) \qquad P_b^* = Q\left(\sqrt{\frac{4\int_0^\tau (s_1(t))^2 dt}{2N_0}}\right)$$

$$P_b^* = Q\left(\sqrt{\frac{\int_0^\tau (s_1(t) + s_1(t))^2 dt}{2N_0}}\right) \qquad \boldsymbol{P_b^* = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

Probability of error decreases as the signal to noise ratio increases